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Finite Element Formulation of Poro-Elasticity Suitable for Large Deformation Dynamic Analysis

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We use theory of mixtures to model the saturated soil as a two-phase medium composed of solid grains and fluids. By applying the conservation laws of momentum and mass we obtain the governing coupled equations. By ignoring the relative acceleration of the fluid phase to that of the solid phase we obtain the simplified ***u***-*p* formulation. We perform the time integration using the Newmark method. To incorporate the finite deformation effects in the context of the ***u***-*p* formulation we consider a compressible Neo-Hookean hyperelastic model describing the constitutive behavior of the porous matrix. We adopt a Lagrangian point of view by integrating the balance equations over the reference configuration domain; however, we represent the constitutive model for fluid flow by a generalized Darcy's law formulated with respect to the current configuration. Fluid compressibility is considered in the finite deformation model based on logarithmic volumetric strain. A numerical example is presented to show the performance of the finite element model.

INTRODUCTION

Porous materials are defined as materials with an internal structure. They comprise a solid phase and closed and open pores. The solid phase is usually referred to as matrix or skeleton. The pores may be filled with one or more kinds of fluids or gas. Soil, rock, concrete are some of the most commonplace porous media. The mechanics of porous media is of utmost interest in many disciplines such as geotechnical engineering, earthquake engineering, geophysics, petroleum engineering, biomechanics, physical chemistry, agricultural engineering, and materials science.

The motivation behind this research is the formulation of a mathematical model characterizing the behavior of fully saturated soil media during dynamic excitation. The challenge lies in the transient behavior of this material as the soil matrix deforms. Such solid deformation is generally accompanied by transient flow of fluids through and across the open pore spaces. Furthermore, large deformation of the soil matrix gives rise to second-order geometric effects not accounted for by the linear theory. Finally, the presence of inertia loads in both the solid and fluid phases makes the solution of the coupled phenomena computationally demanding. To the knowledge of the authors there is currently no systematic way of treating geometric nonlinearity in the context of dynamic analysis of fully saturated soil media.

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This research presents a mathematical framework for characterizing the response of fully saturated soil media subjected to dynamic excitation. The formulation accounts for transient fluid diffusion and finite deformation effects. A nonlinear Neo-Hookean hyperelastic constitutive model is implemented. However, the formulation is intended to serve as a foundation for more advanced computational modeling platforms that may also account for plasticity. Besides, the research methodology and results are general and are applicable to other porous media. By using different material subroutines, the method can be used for geotechnical earthquake engineering applications, to study the behavior of tissues in human body, and to investigate the mechanical properties of the human cornea, among others.

METHODOLOGY

Two sources of nonlinearity exist in the analysis of porous continua, namely, material nonlinearity and geometric nonlinearity. The strong material nonlinearity exhibited by most soil under earthquake loading conditions is well known. A variety of numerical models have been proposed in the last twenty years to capture the relationship of the ground motion response with the input bedrock earthquake excitation, considering nonlinear soil behavior. Finite deformation is another important source of nonlinearity which has not received much attention. Geometric nonlinearity is important in the area of stability analysis, liquefaction analysis, and any situation where the strain level is high.

To address this problem, wave propagation and diffusion effects are coupled through a two-phase theory of mixtures. The governing equations describing the coupling effects of the solid phase and the fluid phase are based on a $\boldsymbol{u}$-p formulation, where $\boldsymbol{u}$ is the solid displacement, p is the pore water pressure. The governing equations are derived from balance of mass and balance of momentum for the overall mixture and for each phase, respectively. These field equations are used to develop the variational forms for the finite element (FE) implementation. A general nonlinear FE framework is then written. The code is used to simulate the dynamic response of saturated soils; however, it can also be applied to more general dynamic FE problems. A hyperelastic soil constitutive model is implemented for finite strain problems. Other materials such as those encountered in biomechanics, or elastoplastic soil constitutive models can also be readily used within this FE framework. In this FE program, a Q9P4 (quadrilateral in displacement and bilinear in pore pressure) element is implemented to simulate plane strain problems in both the small deformation and the finite deformation regimes.

BRIEF BACKGROUND LITERATURE

Theory of porous media has been of great interest to researchers for a considerable time. Biot (1941;1956;1962) generalized Terzaghi's theory of consolidation and developed a two-phase coupled fluid-solid mixture theory for the quasi-static and dynamics analyses. Since then, the governing equations for porous materials have been formulated by many researchers within the framework of linear elasticity. Modern mixture theories were developed by Bowen (1980;1982). Extensions of Biot's theory into the nonlinear range were formulated by Prevost (1980;1982) and others. de Boer (1996) provided a detailed historical review of the development of theory of porous media.

Borja and Alarcón (1985) and Borja et al. (1998) proposed a mathematical framework for finite strain consolidation, where the motion of the solid phase alone was followed. The generalized Darcy's law describes the relative motion of the fluid phase with respect to the solid phase. However their analysis is limited to quasi-static problems, and they assumed that both soil particles and fluids are incompressible. Armero (1999) formulated a multiplicative elastoplastic model in the quasi-static range. Larsson and Larsson (2002) presented a thermodynamic formulation. However, their formulation is still limited to quasi-static analysis. This review is by no means complete, and at the time of writing of this paper, more works are still coming out in the literature. The pertinent details of the model formulation and implementation presented in this paper are described in great length in an upcoming paper by Li, Borja and Regueiro (2004) entitled "Dynamics of porous media at finite strain."

BALANCE LAWS

Let $\phi_t : B \to R^{n_{\rm sd}}$ be the motion, or set of configurations of a fluid saturated simple porous boundary $B \subset R^{n_{\rm sd}}$ and let U be an arbitrary open set with piecewise C^1 boundary condition, such that $U \subset B$. For each material point X in B, we associate Lagrangian displacement, velocity and acceleration fields $\boldsymbol{u}(X,t)$, $\boldsymbol{v}(X,t)$, and $\boldsymbol{a}(X,t)$, where t is time, such that

$$\boldsymbol{u}(X,t) = \boldsymbol{x} - \boldsymbol{X}; \qquad \boldsymbol{v}(X,t) = \frac{\partial \phi(X,t)}{\partial t}; \qquad \boldsymbol{a}(X,t) = \frac{\partial^2 \phi(X,t)}{\partial t^2}, \tag{1}$$

where $\boldsymbol{x} = \phi(X,t)$, and $\boldsymbol{X}$ are the positions of the material point X in the current and reference configurations, respectively. At any spatial point $\boldsymbol{x}$ now occupied by X, we also associate fluid particles that completely fill up the voids of X, with Eulerian velocity in the presence of diffusion given by

$$\boldsymbol{v}_{\rm f} = \boldsymbol{v}_{\rm f}(\boldsymbol{x},t) \neq \boldsymbol{v}(X,t). \tag{2}$$

If $\boldsymbol{v}_{\rm f} = \boldsymbol{v}$, then the fluid and solid move together as one body, leading to a locally undrained motion. In the following derivation, we use $\phi_t(U)$ to represent the deformed body, B to represent the soil body in its original configuration, and $\partial\phi_t(U)$ and ∂B to represent the boundaries of the body in the deformed and original configurations, respectively.

BALANCE OF MASS

Let $\rho_{\rm s}$ be the intrinsic mass density of the solid grains, $\rho_{\rm f}$ the intrinsic mass density of the fluid phase, and φ the porosity of the soil. Balance of mass for the solid phase reads

$$-\frac{{\rm d}\varphi}{{\rm d}t} + \frac{1-\varphi}{K_{\rm s}}\frac{{\rm d}p_{\rm s}}{{\rm d}t} + (1-\varphi)\,{\rm div}\,\boldsymbol{v} = 0, \tag{3}$$

where $K_{\rm s}$ is the intrinsic bulk modulus of the solid grains, $p_{\rm s}$ is the intrinsic pressure in the solid grains, and ${\rm d}(\cdot)/{\rm d}t$ a material time derivative operator following the motion of the solid phase. Balance of mass for the fluid phase writes

$$\frac{{\rm d}\varphi}{{\rm d}t} + \frac{\varphi}{K_{\rm f}}\frac{{\rm d}p_{\rm f}}{{\rm d}t} + \varphi\,{\rm div}\,\boldsymbol{v} = -\frac{1}{\rho_{\rm f}}{\rm div}\,\boldsymbol{q}, \tag{4}$$

where K_{f} is the intrinsic bulk modulus of the fluid phase, p_{f} is the intrinsic pressure in the fluid phase, and $\boldsymbol{q}$ is the Eulerian relative flow vector, given explicitly by

$$\boldsymbol{q} = \varphi\rho_{\mathrm{f}}(\boldsymbol{v}_{\mathrm{f}} - \boldsymbol{v}). \tag{5}$$

Note that the material time derivative is taken following the motion of the solid phase in both the solid and mass balance equations. Balance of mass for the solid-fluid mixture takes the form

$$\frac{1-\varphi}{K_{\mathrm{s}}}\frac{\mathrm{d}p_{\mathrm{s}}}{\mathrm{d}t} + \frac{\varphi}{K_{\mathrm{f}}}\frac{\mathrm{d}p_{\mathrm{f}}}{\mathrm{d}t} + \operatorname{div}\boldsymbol{v} = -\frac{1}{\rho_{\mathrm{f}}}\operatorname{div}\boldsymbol{q}, \tag{6}$$

BALANCE OF LINEAR MOMENTUM

We denote by $\boldsymbol{\sigma}^{\mathrm{s}}$ and $\boldsymbol{\sigma}^{\mathrm{f}}$ the Cauchy partial stress tensors arising from the solid and fluid phase stresses, respectively. The Cauchy total stress tensor is obtained from the sum

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{s}} + \boldsymbol{\sigma}^{\mathrm{s}}. \tag{7}$$

For the solid phase the linear momentum balance equation in the absence of momentum supplies due to chemical reactions with the fluid takes the form

$$\rho_{\mathrm{s}}(1-\varphi)\mathbf{g} + \boldsymbol{h}^{\mathrm{s}} + \operatorname{div}\boldsymbol{\sigma}^{\mathrm{s}} = \rho_{\mathrm{s}}(1-\varphi)\boldsymbol{a}, \tag{8}$$

where $\mathbf{g}$ is the vector of gravity acceleration, and $\boldsymbol{h}^{\mathrm{s}}$ is the flow-induced body force arising from the frictional drag of the fluid phase on the solid matrix. Similarly, for the fluid phase the linear momentum balance equation can be written as follows,

$$\rho_{\mathrm{f}}\varphi\mathbf{g} + \boldsymbol{h}^{\mathrm{f}} + \operatorname{div}\boldsymbol{\sigma}^{\mathrm{f}} = \rho_{\mathrm{f}}\varphi\boldsymbol{a}^{\mathrm{f}}, \tag{9}$$

where $\boldsymbol{h}^{\mathrm{f}}$ is the reactive body force exerted by the solid matrix on the fluid phase. Note that since $\boldsymbol{h}^{\mathrm{s}}$ and $\boldsymbol{h}^{\mathrm{f}}$ are internal forces that naturally will not affect the mixture as a whole, so $\boldsymbol{h}^{\mathrm{s}} + \boldsymbol{h}^{\mathrm{s}} = \mathbf{0}$. In the ***u***-*p* formulation, we assume the material acceleration of the fluid phase relative to that of the solid phase is negligible, i.e., $\boldsymbol{a}^{\mathrm{f}} \approx \boldsymbol{a}$. Adding equations (8) and (9) gives

$$\rho\mathbf{g} + \boldsymbol{h}^{\mathrm{f}} + \operatorname{div}\boldsymbol{\sigma} = \rho\boldsymbol{a}, \tag{10}$$

where $\rho = \rho_{\mathrm{s}}(1-\varphi) + \rho_{\mathrm{f}}\varphi$ represents the total saturated mass density of the mixture.

CONSTITUTIVE EQUATIONS

Instead of expressing the total Cauchy stress tensor in terms of partial stresses, we write it as the sum of effective stresses and pore pressures, i.e.,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}' - p_{\mathrm{f}}\mathbf{1}, \tag{11}$$

where $\mathbf{1}$ is the second-order identity tensor. The negative sign follows from continuum mechanics convention of a positive sign for tensile normal stresses. Equivalently, we write the effective stress equation in terms of the symmetric second Piola-Kirchhoff stress tensors,

$$\boldsymbol{S} = \boldsymbol{S}' - p_{\mathrm{f}} \boldsymbol{C}^{-1}, \tag{12}$$

where $\boldsymbol{C}$ is the right Cauchy-Green deformation tensor.

Next we decompose the effective stress tensor $\boldsymbol{S}'$ into inviscid and viscous parts,

$$\boldsymbol{S}' = \boldsymbol{S}'_{\mathrm{inv}} + \boldsymbol{S}'_{\mathrm{vis}}. \tag{13}$$

For the inviscid part we consider a compressible neo-Hookean hyperelastic material based on a stored energy function proposed by Bonet and Wood (1997),

$$\Psi(X,t) = \frac{\mu}{2}(\mathrm{tr}\,\boldsymbol{C} - 3) - \mu \ln J + \frac{\lambda}{2}(\ln J)^2, \tag{14}$$

where λ and μ are the Lamé constants, and J is the Jacobian of the solid motion. This stored energy function yields the following form of the inviscid stress,

$$\boldsymbol{S}'_{\mathrm{inv}} = \mu \mathbf{1} + (\lambda \ln J - \mu)\boldsymbol{C}^{-1}. \tag{15}$$

For the viscous part we consider a Kelvin solid and postulate stress of the form

$$\boldsymbol{S}'_{\mathrm{vis}} = \frac{\alpha}{2} \mathbf{C} : \dot{\boldsymbol{C}}, \tag{16}$$

where α is a parameter reflecting the viscous damping characteristics of the solid matrix, and

$$\mathbf{C} = 4 \frac{\partial^2 \Psi}{\partial \boldsymbol{C} \otimes \partial \boldsymbol{C}} \tag{17}$$

is the second tangential elasticity tensor.

For fluid flow in the dynamic regime the constitutive equation relates the internal body force vector $\boldsymbol{h}^{\mathrm{f}}$ to the Eulerian relative flow vector $\boldsymbol{q}$, where

$$\boldsymbol{h}^{\mathrm{f}} = \varphi \mathrm{g} \boldsymbol{k}^{-1} \cdot \boldsymbol{q}, \tag{18}$$

where g is the gravity acceleration constant and $\boldsymbol{k}$ is the permeability tensor (assumed symmetric and positive-definite). Imposing balance of momentum for the fluid phase thus gives the generalized Darcy's law in the dynamic regime, given by

$$\boldsymbol{q} = \rho_{\mathrm{f}} \boldsymbol{k} \cdot \Big[\frac{1}{\varphi \rho_{\mathrm{f}} \mathrm{g}} \mathrm{grad}\Big(\frac{\varphi p_{\mathrm{f}}}{J} \Big) + \frac{\boldsymbol{a} - \mathbf{g}}{\mathrm{g}} \Big]. \tag{19}$$

Implied in the above equation is the assumption $\boldsymbol{a}^{\mathrm{f}} \approx \boldsymbol{a}$, allowing the problem to be formulated in $\boldsymbol{u}$-p form.

COMPUTATIONAL ASPECTS

The $\boldsymbol{u}$-p formulation requires $\boldsymbol{u}$ and p to be defined from sets of trial functions that are H^1 in the sense that they have square-integrable first derivatives, and that they satisfy essential boundary conditions. The formulation also requires that weighting functions are also H^1 and vanish on the Dirichlet boundaries. The finite element formulation for the problem at hand is described by Li, Borja and Regueiro (2004). Following the standard finite element formulation, element shape functions are introduced to interpolate the solid phase displacement and pore pressure fields. Here, in order to avoid element locking in the nearly incompressible range, we require the shape functions for the pore pressure field to be one order lower than the shape functions used for the displacement field. The matrix equations are then time-integrated via Newmark method, and solved iteratively using Newton's method.

NUMERICAL EXAMPLE

We consider a saturated porous foundation supporting a vertically vibrating strip footing. The footing load (in MPa) is given by the harmonic function $w(t) = 3 - 3\cos\omega t$, where $\omega = 100$ rad/s is the circular frequency. The footing is 2m wide, and the porous foundation block is 20m wide and 10m deep. Figure 1 shows the finite element mesh; the left vertical boundary is the plane of symmetry, and hence only the right half of the region is modeled. The boundary conditions on the middle line of the block are applied via horizontal rollers. The upper boundary is free. The left, right and bottom boundaries are supported, and no drainage is allowed. The material parameters are taken as: Lamé constants (in MPa) $\lambda = 8.4$ and $\mu = 5.6$; initial porosity is $\varphi_0 = 0.33$; reference intrinsic mass densities (in kg/cu.cm.) $\rho_{\mathrm{s}0} = 2500$ and $\rho_{\mathrm{f}0} = 1000$; fluid bulk stiffness $K_{\mathrm{f}} = 2.2 \times 10^4$ MPa; hydraulic conductivity $\boldsymbol{k} = \kappa \mathbf{1}$ cm/s (isotropic), where κ varies from 0.0001 m/s to 0.1 m/s; and solid matrix damping coefficient $\alpha = 0.02$ s. The time step is taken as $\Delta t = 0.01$ s.

Figure 2 shows the vertical displacements of node D, located directly below the center of the footing, corresponding to $\kappa = 0.0001$ m/s as predicted by the small and finite deformation analyses, respectively. We see that the small deformation solution overestimates settlement, a typical result considering the nature of the formulation. Figure 3 shows the pore pressure at node E and demonstrates a reverse trend, i.e., the small deformation underestimates pore pressure. Hence, for a given imposed displacement, the error of the small deformation solution has a compounding effect in terms of pore pressure prediction.

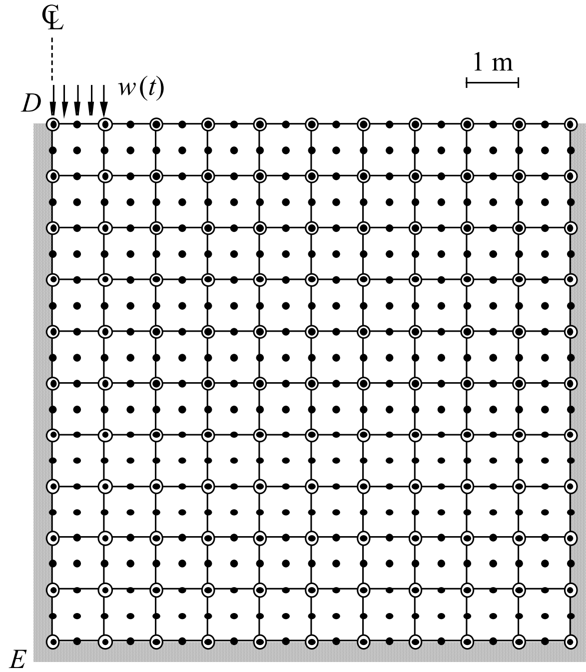

Figure 1. Strip footing under harmonic loading. Uniform pressure $w(t)$ is a harmonic load; vertical sides and bottom boundary are impervious and on roller supports, upper side is a drainage boundary and free (Figure reprinted from Li, Borja and Regueiro, 2004).

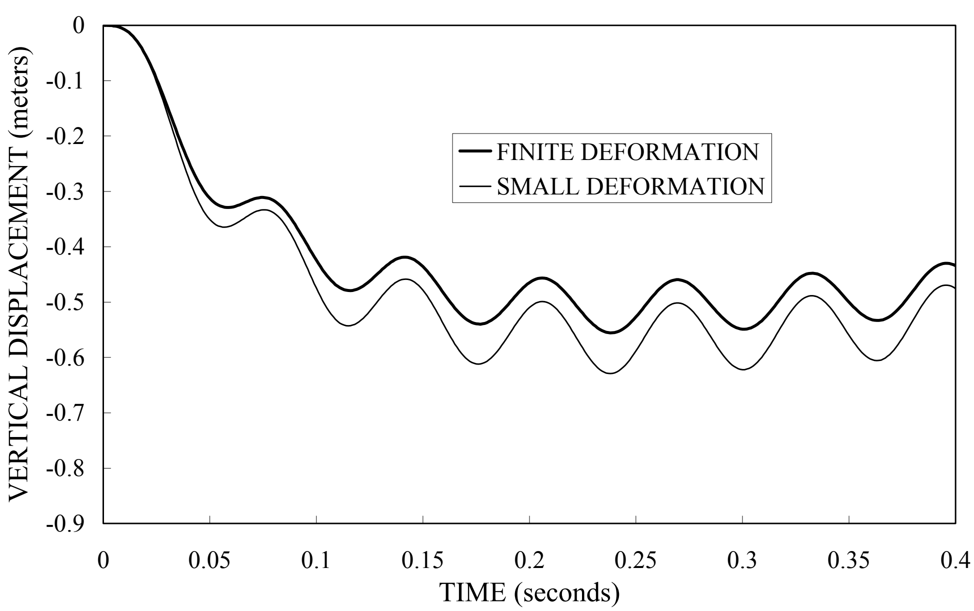

Figure 2. Strip footing under harmonic loading: vertical displacement-time histories of node D at κ = 0.001 m/s (Figure reprinted from Li, Borja and Regueiro, 2004).

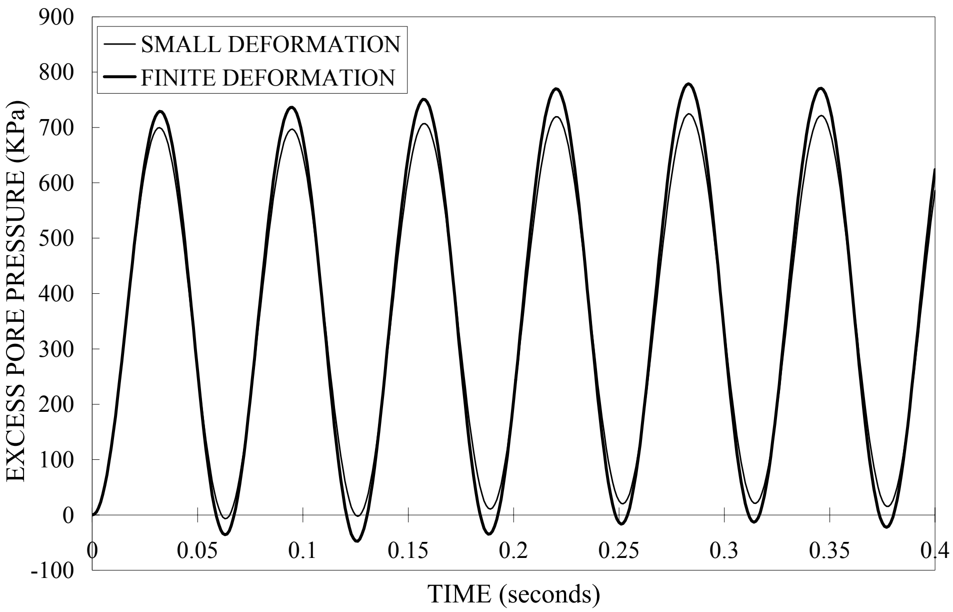

Figure 3. Strip footing under harmonic loading: pore pressure-time histories of node E at $\kappa = 0.001$ m/s (Figure reprinted from Li, Borja and Regueiro, 2004).

SUMMARY AND CONCLUSIONS

A numerical model for nonlinear transient response analyses of fully saturated soil is developed in the framework of $\boldsymbol{u}$-p finite element formulation for finite deformation analyses. The model is implemented into a finite element code. Using theory of mixtures the wave propagation and diffusion phenomena are coupled. The soil is modeled as two-phase continua with interaction between the solid phase and the fluid phase. The governing equations of the coupled soil-fluid mixture are composed of balance of mass and balance of linear momentum. A hyperelastic constitutive model is used for the finite deformation analyses.

A finite deformation formulation is necessary to accurately predict the transient response of saturated porous media at large strains. Geometrically linear models are not suitable for this purpose since they do not account for the evolving configuration and finite rotation that could have first-order effects on the predicted responses. A specific application example where the proposed finite deformation formulation has been noted to be most useful is the prediction of the liquefaction susceptibility of saturated granular soils, since by neglecting the geometric nonlinearity the liquefaction potential of these materials could be severely underestimated. Other application areas abound in the fields of biomechanics and materials science, among many others, where the underlying physics of porous materials is described by multiphase mechanics.

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